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On a direct approach to quasideterminant solutions of a noncommutative KP equation

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Abstract

A noncommutative version of the KP equation and two families of its solutions expressed as quasideterminants are discussed. The origin of these solutions is explained by means of Darboux and binary Darboux transformations. Additionally, it is shown that these solutions may also be verified directly. This approach is reminiscent of the wronskian technique used for the Hirota bilinear form of the regular, commutative KP equation but, in the noncommutative case, no bilinearizing transformation is available.

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1. Introduction

There has been recent interest in a noncommutative version of the Kadomtsev–Petviashvili equation (ncKP) [1–12]

$$(v_t + v_{xxx} + 3v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = 0. \tag{1}$$

This equation can be obtained via the compatibility of the same Lax pair (7)–(8) as is used in the commutative case, but it is not assumed that v and its derivatives commute. In the case that variables do commute, we may differentiate (1) with respect to x and set $v_x = u$ to obtain the well-known (commuting) KP equation

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0. \tag{2}$$

In most of the recent work on ncKP the noncommutativity arises because of a quantization of the phase space in which independent variables do not commute and the (commutative) product of real- or complex-valued functions of these are replaced by the associative but noncommutative Moyal star product. This approach is useful from the point of view of interpreting solutions as they can be expressed in terms of standard functions. It is however conceptually quite difficult because of the noncommutativity of the independent variables.

In this present paper we will not specify the nature of the noncommutativity and the results we present are valid not only in the star product case but also for, for example, the matrix or quarternion versions of the KP equation. This is in the spirit of the work by Etingof, Gelfand and Retakh [13] in which solutions of the ncKP equation were found in terms of quasideterminants [14, 15] using a noncommutative version of Gelfand–Dickey theory. Very recently, Hamanaka [16] has used this form of solution to obtain the soliton solutions of ncKP in the Moyal product case.

We will consider two types of quasideterminant solutions of ncKP. One is equivalent to those found in [13] which we will call *quasiwronskians*. We will also consider a new type of quasideterminant solution which we term a *quasigrammian*. These two types of solution are each constructed by iterating Darboux transformations—the quasiwronskians using a standard Darboux transformation and the quasigrammians using the related binary Darboux transformation. The connection between Darboux transformations for a matrix Schrödinger equation and quasideterminants was also investigated in [17].

We will then show that, in fact, these solutions can be verified by direct substitution. This sort of direct approach is very widely studied in the commutative case (see [18] for the first results for the KP case, and [19] for a discussion of many other examples). In these cases one first makes a change of dependent variable which converts the nonlinear equation to Hirota bilinear form. For the KP equation one writes $u = 2(\log\tau)_{xx}$ and then (2) is converted to Hirota form, a homogeneously quadratic differential equation in τ. A solution τ in the form of a determinant may be verified by recognizing the Hirota form as a determinantal identity such as a Plücker relation or Jacobi identity.

In contrast, the central role of the τ-function (a determinant) in the commutative case is taken by the quasideterminant in the noncommutative case. In this case there is no bilinearizing change of variables since v is expressed directly as a quasideterminant. Also, remarkably given what happens in the commutative case, no use is made of special identities in verifying the quasideterminant solutions of ncKP. Some remarks on the reason for this are given later. Paradoxically, direct verification of solutions in the noncommutative case is in a number of respects easier than in the commutative case. However, this state of affairs seems to be particular to ncKP. In other examples we have considered [20–22] a change of variables and use of quasideterminant identities are necessary to achieve direct verification of solutions.

2. Preliminaries

In this short section we recall some of the key elementary properties of quasideterminants. The reader is referred to the original papers [14, 15] for a more detailed and general treatment.

2.1. Quasideterminants

An $n \times n$ matrix A over a ring $\mathcal{R}$ (non-commutative, in general) has n^2 *quasideterminants* written as $|A|_{i,j}$ for $i, j = 1, \ldots, n$, which are also elements of $\mathcal{R}$. They are defined recursively by

$$|A|_{i,j} = a_{i,j} - r_i^j (A^{i,j})^{-1} c_j^i, \qquad A^{-1} = \left(|A|_{j,i}^{-1}\right)_{i,j=1,\ldots,n}.$$

In the above r_i^j represents the ith row of A with the jth element removed, c_j^i the jth column with the ith element removed and $A^{i,j}$ the submatrix obtained by removing the ith row and the jth column from A. Quasideterminants can also denoted as shown below by boxing the entry about which the expansion is made:

$$|A|_{i,j} = \begin{vmatrix} A^{i,j} & c_j^i \\ r_i^j & \boxed{a_{i,j}} \end{vmatrix}.$$

The case $n = 1$ is rather trivial; let $A = (a)$, say, and then there is one quasideterminant $|A|_{1,1} = |\boxed{a}| = a$. For $n = 2$, let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then there are four quasideterminants,

$$|A|_{1,1} = \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c, \qquad |A|_{1,2} = \begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d,$$

$$|A|_{2,1} = \begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a, \qquad |A|_{2,2} = \begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b.$$

From this we can obtain the matrix inverse,

$$A^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix},$$

which is then used in the definition of the nine quasideterminants of a 3×3 matrix. Note that if the entries in A commute, the above becomes the familiar formula for the inverse of a 2×2 matrix with entries expressed as ratios of determinants. Indeed this is true for any size of square matrix; if the entries in A commute then

$$|A|_{i,j} = (-1)^{i+j} \frac{\det(A)}{\det(A^{i,j})}. \tag{3}$$

In this paper we will consider only quasideterminants that are expanded about a term in the last column, most usually the last entry. For a block matrix

$$\begin{pmatrix} A & B \\ C & d \end{pmatrix}$$

where $d \in \mathcal{R}$, A is a square matrix over $\mathcal{R}$ of arbitrary size and B, C are column and row vectors over $\mathcal{R}$ of compatible lengths, we have

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B.$$

2.2. Invariance under row and column operations

The quasideterminants of a matrix have invariance properties similar to those of determinants under elementary row and column operations applied to the matrix. Consider the following quasideterminant of an $n \times n$ matrix:

$$\left| \begin{pmatrix} E & 0 \\ F & g \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \right|_{n,n} = \begin{vmatrix} EA & EB \\ FA + gC & FB + gd \end{vmatrix}_{n,n} = g(d - CA^{-1}B) = g \begin{vmatrix} A & B \\ C & d \end{vmatrix}_{n,n}. \tag{4}$$

The above formula can be used to understand the effect on a quasideterminant of certain elementary row operations involving multiplication on the left. This formula excludes those operations which add left-multiples of the row containing the expansion point to other rows since there is no simple way to describe the effect of these operations. For the allowed operations, however, the results can be easily described; left-multiplying the row containing the expansion point by g has the effect of left-multiplying the quasideterminant by g and all other operations leave the quasideterminant unchanged. There is analogous invariance under column operations involving multiplication on the right.

2.3. Noncommutative Jacobi identity

There is a quasideterminant version of Jacobi's identity for determinants, called the noncommutative Sylvester's theorem by Gelfand and Retakh [14]. The simplest version of this identity is given by

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}. \tag{5}$$

3. Solutions obtained using Darboux transformations

3.1. Darboux transformation

Let L be an operator covariant under the Darboux transformation $G_\theta = \theta\partial_x\theta^{-1} = \partial_x - \theta_x\theta^{-1}$, where θ is an eigenfunction of L (i.e. $L(\theta) = 0$). Let $\theta_i, i = 1, \ldots, n$ be a particular set of eigenfunctions and introduce the notation $\Theta = (\theta_1, \ldots, \theta_n)$ and $\widehat{\Theta} = \left(\theta_j^{(i-1)}\right)_{i,j=1,\ldots,n}$, the $n \times n$ wronskian matrix of $\theta_1, \ldots, \theta_n$, where $^{(k)}$ denotes the kth x-derivative.

Let $\phi_{[1]} = \phi$ be a general eigenfunction of $L_{[1]} = L$ and $\theta_{[1]} = \theta_1$. Then $\phi_{[2]} := G_{\theta_{[1]}}(\phi_{[1]})$ and $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$ are eigenfunctions for $L_{[2]} = G_{\theta_{[1]}} L_{[1]} G_{\theta_{[1]}}^{-1}$. In general, for $n \geqslant 1$ define the nth Darboux transform of ϕ by

$$\phi_{[n+1]} = \phi_{[n]}^{(1)} - \theta_{[n]}^{(1)}\theta_{[n]}^{-1}\phi_{[n]},$$

in which

$$\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}.$$

It is known that [15, 17]

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}. \tag{6}$$

The Lax pair for the ncKP equation (1) is

$$L = \partial_x^2 + v_x - \partial_y, \tag{7}$$

$$M = 4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y + \partial_t. \tag{8}$$

Both L and M are covariant with respect to the above Darboux transformation. Moreover, it is straightforward to calculate that the effect of the Darboux transformation

$$\tilde{L} = G_\theta L G_\theta^{-1}, \qquad \tilde{M} = G_\theta M G_\theta^{-1},$$

is that $\tilde{v} = v + 2\theta_x\theta^{-1}$. Thus after n Darboux transformations we obtain

$$v_{[n+1]} = v + 2\sum_{i=1}^{n} \theta_{[i],x}\theta_{[i]}^{-1}, \tag{9}$$

which describes a class of solutions of ncKP. An analogous formula is obtained using a noncommutative version of Gelfand–Dickey theory in [13]. Further, it may be proved by induction from (6), making use of an identity of the form (5), that

$$\sum_{i=1}^{n}\theta_{[i],x}\theta_{[i]}^{-1} = -\begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}. \tag{10}$$

Thus, using Darboux transformations, we have obtained a formula for solutions $v_{[n+1]}$ of ncKP expressed in terms of a known solution v and a single wronskian-like quasideterminant,

$$v_{[n+1]} = v - 2\begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}. \tag{11}$$

3.2. Binary Darboux transformation

To define a binary Darboux transformation one needs to consider the adjoint Lax pair. The notion of adjoint is easily extended from the familiar matrix situation to any ring $\mathcal{R}$ (see [23]); suppose that for each $a \in \mathcal{R}$, there exists $a^\dagger \in \mathcal{R}$, for any derivative ∂ acting on $\mathcal{R}$, $\partial^\dagger = -\partial$ and for any product AB of elements of, or operators on $\mathcal{R}$, $(AB)^\dagger = B^\dagger A^\dagger$. Accordingly, the adjoint Lax pair is

$$L^\dagger = \partial_x^2 + v_x^\dagger + \partial_y, \tag{12}$$

$$M^\dagger = -4\partial_x^3 - 6v_x^\dagger\partial_x - 3v_{xx}^\dagger + 3v_y^\dagger - \partial_t. \tag{13}$$

Following the standard construction of a binary Darboux transformation (see [24, 25]), one introduces a potential $\Omega(\phi,\psi)$ satisfying

$$\begin{aligned} &\Omega(\phi,\psi)_x = \psi^\dagger\phi, \qquad \Omega(\phi,\psi)_y = \psi^\dagger\phi_x - \psi_x^\dagger\phi, \\ &\Omega(\phi,\psi)_t = -4\left(\psi^\dagger\phi_{xx} - \psi_x^\dagger\phi_x + \psi_{xx}^\dagger\phi\right) - 6\psi^\dagger v_x\phi. \end{aligned} \tag{14}$$

The parts of this definition are compatible when $L(\phi) = M(\phi) = 0$ and $L^\dagger(\psi) = M^\dagger(\psi) = 0$. More generally, we can define $\Omega(\Phi,\Psi)$ for any row vectors Φ and Ψ such that $L(\Phi) = M(\Phi) = 0$ and $L^\dagger(\Psi) = M^\dagger(\Psi) = 0$. If Φ is an n-vector and Ψ is an m-vector then $\Omega(\Phi,\Psi)$ is an $m \times n$ matrix. The adjoint of a $p \times q$ matrix $A = (a_{i,j})$ over $\mathcal{R}$ has an obvious meaning, it is the $q \times p$ matrix $A^\dagger = \left(a_{j,i}^\dagger\right)$.

A binary Darboux transformation is then defined by

$$\phi_{[n+1]} = \phi_{[n]} - \theta_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-1}\Omega(\phi_{[n]},\rho_{[n]})$$

and

$$\psi_{[n+1]} = \psi_{[n]} - \rho_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-\dagger}\Omega(\theta_{[n]},\psi_{[n]})^\dagger,$$

where

$$\theta_{[n]} = \phi_{[n]}|_{\phi\to\theta_n}, \qquad \rho_{[n]} = \psi_{[n]}|_{\psi\to\rho_n}.$$

Using the notation $\Theta = (\theta_1, \ldots, \theta_n)$ and $\mathrm{P} = (\rho_1, \ldots, \rho_n)$ it is easy to prove by induction that, for $n \geqslant 1$,

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix}, \tag{15}$$

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P})^\dagger & \Omega(\Theta, \psi)^\dagger \\ \mathrm{P} & \boxed{\psi} \end{vmatrix}, \tag{16}$$

and

$$\Omega(\phi_{[n+1]}, \psi_{[n+1]}) = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Omega(\Theta, \psi) & \boxed{\Omega(\phi, \psi)} \end{vmatrix}. \tag{17}$$

As for the effect of these transformation on the Lax pair, a transformation by θ, ρ gives new coefficients defined in terms of

$$\hat{v} = v + 2\theta\Omega(\theta, \rho)^{-1}\rho^\dagger.$$

Thus after n binary Darboux transformations we obtain

$$v_{[n+1]} = v + 2\sum_{k=1}^{n} \theta_{[k]}\Omega(\theta_{[k]}, \rho_{[k]})^{-1}\rho_{[k]}^\dagger, \tag{18}$$

and this may be reexpressed in terms of a single quasideterminant as

$$v_{[n+1]} = v - 2\begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & \boxed{0} \end{vmatrix}. \tag{19}$$

In this way we have obtained a second expression for solutions of the ncKP equation, this time in terms of grammian-type quasideterminants.

In the following sections, we will show that these two quasideterminant solutions (wronskian-type and grammian-type) of the ncKP equation may also be verified by direct calculation in the spirit of Hirota's direct method.

4. Derivatives of a quasideterminant

We can derive a rather appealing formula for derivatives of a quasideterminant which resembles the formula for derivatives of a normal determinant. Consider the derivative

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B' \tag{20}$$

where A is an $n \times n$ matrix, C is a row vector and B a column vector. If the matrix A has the grammian-like property that its derivative is a scalar product

$$A' = \sum_{i=1}^{k} E_i F_i,$$

where E_i (F_i) are column (row) vectors of appropriate length, then the third term on the RHS of (20) can be factorized as a product of quasideterminants, i.e.,

$$\begin{aligned} \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' &= d' - C'A^{-1}B + \sum_{i=1}^{k}(CA^{-1}E_i)(F_iA^{-1}B) - CA^{-1}B' \\ &= \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{0} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{0} \end{vmatrix}. \end{aligned} \tag{21}$$

Even if the matrix A does not have this grammian-like structure then the third term on the RHS of (20) can still be factorized as a product by inserting the $n \times n$ identity matrix expressed in the form

$$I = \sum_{k=1}^{n} e_k e_k^t,$$

where e_k is the n-vector (δ_{ik}) (i.e. a column vector with 1 in the kth row and 0 elsewhere). Let Z^k denote the kth row and Z_k the kth column of a matrix Z. In this way we have

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B + \sum_{k=1}^{n}(CA^{-1}e_k)\big(e_k^t A' A^{-1} B\big) - \sum_{k=1}^{n}(CA^{-1}e_k)e_k^t B'.$$

Note here that we have also introduced this form of the identity element into the last term on the RHS. This gives

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & e_k \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & \boxed{(B^k)'} \end{vmatrix}. \tag{22}$$

In a similar way, by inserting the identity matrix in a different position we have a column version of the derivative formula

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix} \begin{vmatrix} A & B \\ e_k^t & \boxed{0} \end{vmatrix}. \tag{23}$$

4.1. Derivatives of quasiwronskians

In this section we will calculate derivatives of a quasideterminant of the form

$$Q(i,j) = \begin{vmatrix} \widehat{\Theta} & e_{n-j} \\ \Theta^{(n+i)} & \boxed{0} \end{vmatrix}, \tag{24}$$

where, as above, $\widehat{\Theta} = \big(\theta_j^{(i-1)}\big)_{i,j=1,\ldots,n}$ is the $n \times n$ wronskian matrix of $\theta_1, \ldots, \theta_n$. In this definition, i and j are allowed to take any integer values subject to the convention that if $n - j$ lies outside the range $1, 2, \ldots, n$, then $e_{n-j} = 0$ and so $Q(i,j) = 0$. There is an important special case; when $n+i = n-j-1 \in [0, n-1]$, (i.e. $i+j+1 = 0$ and $-n \leqslant i < 0$) we have

$$Q(i,j) = \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n+i)} & 1 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n+i)} & \boxed{0} \end{vmatrix} = \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n+i)} & 1 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ 0 & \boxed{-1} \end{vmatrix} = -1,$$

using the definition of quasideterminants and the invariance properties described in (4). Using the same argument for $n+i \in [0, n-1]$ but $n+i \neq n-j-1$ we see that $Q(i,j) = 0$. Assuming n is arbitrarily large we may summarize these properties of $Q(i,j)$ as

$$Q(i,j) = \begin{cases} -1 & i+j+1 = 0 \\ 0 & (i < 0 \text{ or } j < 0) \quad \text{and} \quad i+j+1 \neq 0. \end{cases} \tag{25}$$

Readers familiar with symmetric functions will recognize this property as analogous to a property of a hook Schur function $s_{(i|j)}$ (see [26, p 47, example 9]).

We shall call this type of quasideterminant a *quasiwronskian*. In the last section (see (9)) we showed by means of Darboux transformations that if v_0 is any given solution of ncKP and Θ an n-row vector of eigenfunctions of L and M given by (7)–(8) then

$$v = v_0 - 2Q(0, 0) \tag{26}$$

also satisfies ncKP. For simplicity we will choose the vacuum solution $v_0 = 0$ but this choice of vacuum is not essential to what follows; the direct verification can be made for arbitrary vacuum but the formulae are rather more complicated.

If we relabel and rescale the independent variables so that $x_1 = x$, $x_2 = y$, $x_3 = -4t$, Θ satisfies the linear equations

$$\Theta_{x_2} = \Theta_{xx}, \qquad \Theta_{x_3} = \Theta_{xxx}. \tag{27}$$

We may also allow Θ to depend on higher variables x_k and impose the natural dependence $\Theta_{x_k} = \Theta_{\underbrace{x\cdots x}_{k\text{ copies}}}$.

Now for any m, using the linear equations for Θ, we have

$$\begin{aligned}\frac{\partial}{\partial x_m}Q(i,j) &= \begin{vmatrix}\widehat{\Theta} & e_{n-j}\\ \Theta^{(n+i+m)} & \boxed{0}\end{vmatrix} + \sum_{k=1}^{n}\begin{vmatrix}\widehat{\Theta} & e_k\\ \Theta^{(n+i)} & \boxed{0}\end{vmatrix}\begin{vmatrix}\widehat{\Theta} & e_{n-j}\\ \Theta^{(k-1+m)} & \boxed{0}\end{vmatrix}\\ &= Q(i+m,j) + \sum_{k=0}^{n-1} Q(i,k)Q(m-1-k,j).\end{aligned} \tag{28}$$

Using conditions (25) the above simplifies considerably and we obtain

$$\frac{\partial}{\partial x_m}Q(i,j) = Q(i+m,j) - Q(i,j+m) + \sum_{k=0}^{m-1} Q(i,k)Q(m-k-1,j). \tag{29}$$

In particular

$$\frac{\partial}{\partial x}Q(i,j) = Q(i+1,j) - Q(i,j+1) + Q(i,0)Q(0,j),$$
$$\frac{\partial}{\partial x_2}Q(i,j) = Q(i+2,j) - Q(i,j+2) + Q(i,1)Q(0,j) + Q(i,0)Q(1,j),$$
$$\frac{\partial}{\partial x_3}Q(i,j) = Q(i+3,j) - Q(i,j+3) + Q(i,2)Q(0,j) + Q(i,1)Q(1,j) + Q(i,0)Q(2,j).$$

Note that these simplified formulae (29) are only valid for sufficiently large n. For smaller n we should use (28) directly.

4.2. Derivatives of quasigrammians

Let us define

$$R(i,j) = (-1)^j\begin{vmatrix}\Omega(\Theta, \mathrm{P}) & \mathrm{P}^{\dagger(j)}\\ \Theta^{(i)} & \boxed{0}\end{vmatrix},$$

and call this type of quasideterminant a *quasigrammian*. As we have seen in (19), solutions obtained by binary Darboux transformation are of the form $v = v_0 - 2R(0, 0)$. As we did in the case of the quasiwronskian-type of solutions we choose $v_0 = 0$ for simplicity. Hence Θ satisfies the same linear equations as before and P, the vector of adjoint eigenfunctions, satisfies

$$\mathrm{P}_{x_2} = -\mathrm{P}_{xx}, \qquad \mathrm{P}_{x_3} = \mathrm{P}_{xxx}.$$

Note that choice of the trivial vacuum is inessential and direct verification can be completed for arbitrary vacuum.

Using (21), derivatives with respect to the x_m can be calculated:

$$\begin{aligned}
\partial_{x_m} R(i,j) &= (-1)^j \begin{vmatrix} \Omega & \mathrm{P}^{\dagger(j)} \\ \Theta^{(i+m)} & \boxed{0} \end{vmatrix} + (-1)^{m+j-1} \begin{vmatrix} \Omega & \mathrm{P}^{\dagger(j+m)} \\ \Theta^{(i)} & \boxed{0} \end{vmatrix} \\
&\quad + \sum_{k=0}^{m-1} \begin{vmatrix} (-1)^{j+k}\Omega & \mathrm{P}^{\dagger(k)} \\ \Theta^{(i)} & \boxed{0} \end{vmatrix} \begin{vmatrix} \Omega & \mathrm{P}^{\dagger(j)} \\ \Theta^{(m-1-k)} & \boxed{0} \end{vmatrix} \\
&= R(i+m,j) - R(i,j+m) + \sum_{k=0}^{m-1} R(i,k)R(m-k-1,j).
\end{aligned}$$

Note here that this final form for a derivative of a quasigrammian corresponds precisely with the formula for the quasiwronskian (see (29)). Thus calculations in the subsequent sections carried out for the quasiwronskian solutions will be equally valid for the quasigrammian solutions.

4.3. The commutative case

In order to better understand the derivative formulae we obtained above, we will assume here that all quantities commute and hence reduce to the familiar case of the commutative KP equation. Using (3), we have

$$Q(0,0) = -\frac{\begin{vmatrix} \Theta \\ \vdots \\ \Theta^{(n-2)} \\ \Theta^{(n)} \end{vmatrix}}{|\widehat{\Theta}|}, \qquad R(0,0) = \frac{\begin{vmatrix} \Omega & \mathrm{P} \\ \Theta & \boxed{0} \end{vmatrix}}{|\Omega|}.$$

It is then simple to show that $u = -2Q(0,0)_x = 2(\log|\widehat{\Theta}|)_{xx}$ and $u = -2R(0,0)_x = 2(\log|\Omega|)_{xx}$ which are the well-known solutions of the standard KP equation in wronskian and grammian form, respectively.

5. The direct approach

Returning to the noncommutative case, we will show directly that

$$v = -2Q(0,0), \qquad \text{or} \qquad v = -2R(0,0), \tag{30}$$

are solutions of the ncKP equation. To carry out this direct verification we first calculate the derivatives of v,

$$\begin{aligned}
v_x &= -2Q(0,0)_x = -2[Q(1,0) - Q(0,1) + Q(0,0)Q(0,0)] \\
v_y &= -2Q(0,0)_y = -2[Q(2,0) - Q(0,2) + Q(0,0)Q(1,0) + Q(0,1)Q(0,0)] \\
v_t &= -2Q(0,0)_t = 8[Q(3,0) - Q(0,3) + Q(0,0)Q(2,0) \\
&\qquad + Q(0,1)Q(1,0) + Q(0,2)Q(0,0)] \\
v_{xx} &= -2[Q(0,2) - 2Q(1,1) + Q(2,0) - 2Q(0,0)Q(0,1) + Q(0,0)Q(1,0) \\
&\qquad - Q(0,1)Q(0,0) + 2Q(1,0)Q(0,0) + 2Q(0,0)Q(0,0)Q(0,0)] \\
v_{yy} &= -2[Q(0,4) - 2Q(2,2) + Q(4,0) \\
&\qquad + Q(0,0)Q(3,0) + Q(0,1)Q(2,0) - Q(0,2)Q(1,0) - Q(0,3)Q(0,0)
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2Q(0,0)Q(1,2) - 2Q(0,1)Q(0,2) + 2Q(2,0)Q(1,0) + 2Q(2,1)Q(0,0)\\
&\quad + 2Q(0,0)Q(1,0)Q(1,0) + 2Q(0,0)Q(1,1)Q(0,0)\\
&\quad + 2Q(0,1)Q(0,0)Q(1,0) + 2Q(0,1)Q(0,1)Q(0,0)]\\
v_{xt} &= 8[Q(0,4) - Q(1,3) - Q(3,1) + Q(4,0)\\
&\quad + Q(0,0)Q(3,0) - Q(0,3)Q(0,0) - Q(0,0)Q(2,1) - Q(0,2)Q(0,1)\\
&\quad - Q(0,1)Q(1,1) - Q(0,0)Q(0,3) + Q(1,0)Q(2,0) + Q(1,1)Q(1,0))\\
&\quad + Q(1,2)Q(0,0)) + Q(3,0)Q(0,0) + Q(0,0)Q(0,0)Q(2,0)\\
&\quad + Q(0,0)Q(0,1)Q(1,0) + Q(0,0)Q(0,2)Q(0,0) + Q(0,0)Q(2,0)Q(0,0)\\
&\quad + Q(0,1)Q(1,0)Q(0,0) + Q(0,2)Q(0,0)Q(0,0)]
\end{aligned}
$$

and v_{xxxx}, which is straightforward but tedious to work out. This sort of calculation can be readily carried out using any computer algebra package that understands, or can be made to understand, noncommutative multiplication. Substituting these into the ncKP equation (1) all terms exactly cancel and the solution is verified. As remarked above, the derivative formulae are the same whether we use the quasiwronskian or the quasigrammian formulation and so the above calculation simultaneously verifies both types of solution.

6. Comparison with the bilinear approach

The direct approach to the determinantal solutions of the commutative KP equation is well known and can be found in many places in the literature (see [18, 19] for example). Here we will compare it to the alternative direct approach we studied above. In Hirota's direct method, one first makes the change of variables

$$u = 2(\log\tau)_{xx}$$

and then rewrites (2) in bilinear form using Hirota derivatives

$$(D_{xt} + D_{xxxx} + 3D_{yy})\tau\cdot\tau = 0, \tag{31}$$

where [19]

$$D_x^m D_y^n \tau\cdot\tau := \frac{\partial^m\partial^n}{\partial a^m\partial b^m}(\tau(x+a, y+b)\tau(x-a, y-b))\Big|_{a,b\to 0}.$$

The next step is to take a possible solution such as a wronskian or grammian determinant and calculate the derivatives with respect to x, y and t. So, for example, for a wronskian solution we would take (see [18] for an explanation of the notation)

$$\tau = |\widehat{\Theta}|,$$

and calculate the derivatives

$$
\begin{aligned}
\tau_x &= (0,\ldots,n-2,n) = \tau_{(1)},\\
\tau_{xx} &= (0,\ldots,n-2,n+1) + (0,\ldots,n-3,n-1,n) = \tau_{(2)} + \tau_{(1^2)},\\
\tau_y &= \tau_{(2)} - \tau_{(1^2)},
\end{aligned}
$$

and so on. Here we use a shorthand partition notation which denotes the extra derivatives added to each row in the wronksian τ.

Substituting these into the left-hand side of (31) we obtain a constant multiple of

$$\tau_{(2^2)}\tau - \tau_{(21)}\tau_{(1)} + \tau_{(2)}\tau_{(1^2)}. \tag{32}$$

While it is initially not obvious that this expression is identically zero, using the Laplace expansion of a $2n \times 2n$ determinant, one verifies that (32) is indeed zero and the verification is complete.

For the noncommutative case the approach is quite similar; however, curiously some of the steps taken in the bilinear approach are not needed. First, we do not need a Cole–Hopf style change of variables, since the solution is expressed directly as a quasiwronskian. Second, once we have substituted the derivatives into the nonlinear equation the resulting expression immediately vanishes without the need to consider any quasideterminant identities.

The fact that no identities are needed is rather unexpected but a closer examination of the derivatives of $Q(0,0)$ in the commutative case is illuminating. When all quantities commute we may use (3) to obtain

$$Q(i,j) = (-1)^{j-1}\frac{\tau_{(i+1,1^j)}}{\tau},$$

and in particular $Q(0,0) = -\tau_{(1)}/\tau = -\tau_x/\tau$. Calculating the t derivative of each side of this gives

$$\begin{aligned}-\tfrac{1}{4}Q(0,0)_t &= Q(3,0) - Q(0,3) + Q(0,0)Q(2,0) + Q(0,1)Q(1,0) + Q(0,2)Q(0,0)\\ &= -\frac{\tau_{(1^4)} + \tau_{(4)}}{\tau} + \frac{\tau_{(1)}\tau_{(3)} - \tau_{(1^2)}\tau_{(2)} + \tau_{(1^3)}\tau_{(1)}}{\tau^2},\end{aligned}$$

whereas

$$-\frac{1}{4}\left(\frac{-\tau_x}{\tau}\right)_t = -\frac{1}{4}\left(-\frac{\tau_{xt}}{\tau} + \frac{\tau_x\tau_t}{\tau^2}\right) = -\frac{\tau_{(1^4)} - \tau_{(2^2)} + \tau_{(4)}}{\tau} + \frac{\tau_{(1)}(\tau_{(3)} - \tau_{(21)} + \tau_{(1^3)})}{\tau^2}.$$

Note that the term $\tau_{(2^2)}$ cannot come from $Q(i,j)$ for any i, j and that the two expression for the derivatives only agree when one makes use of the identity $\tau_{(2^2)}\tau - \tau_{(21)}\tau_{(1)} + \tau_{(2)}\tau_{(1^2)} = 0$. So it seems that, in some sense, the identity used by hand in verifying solutions in the bilinear approach is used automatically as derivatives are calculated in the quasideterminant approach.

7. Conclusions

In this paper we considered two types of quasideterminant solutions of the noncommutative KP equation. As well as showing how they may be constructed by Darboux transformations, they turn out to be ideal for direct verification of the solution and play the same role that the τ-function does in the commutative case.

There are some interesting features to the direct approach using quasideterminants. First, it illustrates that a bilinear form is not needed, and indeed we believe that it does not exist, in the noncommutative case. The second rather surprising feature is that, unlike the commutative case, no identity is needed to complete the verification.

Noncommutative versions of other integrable equations we have studied, a noncommutative Hirota–Miwa equation [20, 22] and modified KP equation [21], also have quasideterminant solutions. However, in these cases, direct verification does require the use of quasideterminant identities of the form (5) and so it seems that the ncKP equation may be exceptional in this respect.

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